or Dirichlet's limit. But if x and ξ are not identical, and if we know their difference with sufficient precision—that is, within a negligible fraction of the wave-length $2\pi c^m$ where the *lower* bound of m is as stated above—nothing is easier than to calculate λ_1 and λ_2, and thus $F_i(x)$. It will be seen that $F_i(x)$, as x increases through ξ, passes quite continuously from $f(\xi-0)$ to $f(\xi+0)$, since λ_1 changes from 1 to 0, and λ_2 from 0 to 1, in a perfectly continuous manner. The Fourier integral $F(x)$ is seen to be continuous however discontinuous $f(x)$ may be. F and f cannot therefore be identical. They are numerically equal in the sense that $F-f$ is infinitesimal compared with f, and that too for every value of x within the specified range of x. It is possible that many ordinary mathematical theorems state only numerical equalities in the above sense, because in the proof of such theorems it is usual to neglect an infinitesimal at some stage of the argument.

XXVII. *On Simultaneous Operational Calculus.*

By Balth. van der Pol, *D.Sc.*, *and* K. F. Niessen, *D.Sc.*[*]

1. According to J. R. Carson [†] Heaviside's operational calculus can be based upon the transformation of a given function $h(x)$ into its "image" $f(p)$, related to the former one by means of

$$f(p)=p\int_0^{\infty} e^{-px}h(x)\,dx.$$

This relation between image and original we write

$$f(p)\doteq h(x).$$

If the image $f(p)$ is known, one can obtain the original function $h(x)$ in the way pointed out by Bromwich [‡], leading to one single function $h(x)$ only on account of a theorem of March [‡].

The number p, whose real part must be positive

$$R(p)>0,$$

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[*] Communicated by the Authors.

[†] J. R. Carson, 'Electric Circuit Theory and the Operational Calculus' (McGraw-Hill, New York, 1926).

[‡] For references about these articles see Balth. van der Pol, Phil. Mag. viii. p. 861 (1929), which will be quoted as I. See also H. M. Macdonald, Math. Soc. of London, xxxv. p. 428 (1902).

is called the parameter which operationally belongs to x, although really $\frac{1}{p}$ (not p) and x are related to one another as image and original :

$$\frac{1}{p} \doteqdot x.$$

We now wish to introduce a simultaneous operational calculus, where two variables x, y are treated operationally by means of two parameters p, q respectively.

$$\left.\begin{array}{l} \frac{1}{p} \doteqdot x, \\ \\ \frac{1}{q} \doteqdot y, \end{array}\right\} \quad \text{or in general} \quad \left\{\begin{array}{l} f_1(p) \doteqdot h_1(x), \\ \\ f_2(q) \doteqdot h_2(y), \end{array}\right.$$

based upon

$$f_1(p) = p\int_0^\infty e^{-px} h_1(x)\, dx,$$

$$f_2(q) = q\int_0^\infty e^{-qy} h_2(y)\, dy.$$

As an example we will determine the original function $H(x, y)$ corresponding to the image:

$$\frac{1}{(p+q)^n} \doteqdot H(x, y).$$

Regarding for an instance q as a constant, we first investigate the interpretation $h(x)$ of

$$f_1(p) = \frac{1}{(p+q)^n} \doteqdot h_1(x).$$

Applying a known theorem (see Carson or van der Pol, *l.c.* (8)), we have

$$\frac{p}{p-c} f_1(p-c) \doteqdot e^{cx} h_1(x),$$

where c is any constant.

Choosing q for c, we get

$$\frac{p}{p-q} \cdot \frac{1}{p^n} \doteqdot e^{qx} h_1(x), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where the sign $\doteqdot$ is related to the transformation $\frac{1}{p} \doteqdot x$.

We might better write

$$h_1(x) = h_q(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

to indicate that there is still the parameter q left. Now we have, keeping q still constant,

$$\frac{p}{p-q} \doteqdot e^{qx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the known theorem, that when

$$f(p) \doteqdot h(x),$$

it follows that

$$\frac{1}{p} f(p) \doteqdot \int_0^x h(x)\, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

it is tacitly understood that

$$\int_0^\infty h(x)\, dx << e^{p\infty}.$$

This is seen by performing the partial integration

$$p\int_0^\infty e^{-px} \int_0^x h(x)\, dx = -\int_0^\infty \int_0^x h(x)\, dx\, d.e^{-px}$$

$$= e^{-p\infty} \int_0^\infty h(x)\, dx + f(p)/p.$$

Therefore the application of (4) to (3) is only permitted in the case

$$p > q,$$

giving

$$\frac{p}{p-q} \frac{1}{p^n} \doteqdot \int_0^x \dots \int_0^x e^{qx} (dx)^n.$$

This multiple integral can be transformed into a single one in a known way, so that

$$\frac{p}{p-q} \frac{1}{p^n} \doteqdot \int_0^x e^{q\xi} \frac{(x-\xi)^{n-1}}{(n-1)\,!} d\xi. \quad . \quad . \quad . \quad . \quad (5)\,*$$

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[*] This transformation follows also directly when the left-hand side of (5) is written as

$$\frac{p}{p-q} \cdot \frac{1}{p^{n-1}} \cdot \frac{1}{p},$$

and is considered as a "produit de composition."

Comparing (5) and (1, 2) we find

$$e^{qx}h_q(x)=\int_0^x e^{q\xi}\frac{(x-\xi)^{n-1}}{(n-1)!}d\xi.$$

Writing $x-\xi=s$,

we obtain $h_q(x)=\dfrac{1}{(n-1)!}\displaystyle\int_0^x e^{-qs}s^{n-1}\,ds,$

an incomplete gamma-function.

In order to find the required function $H(x, y)$ we have to transform $h_q(x)$ as a function of q operationally into a function of y:

$$f(q)=\frac{1}{(n-1)!}\int_0^x e^{-qs}s^{n-1}ds \doteqdot H(x, y). \quad . \quad . \quad (6)$$

With reference to I. form. (9) we have

$$e^{-sq} \doteqdot F_s(y)=\left\{\begin{matrix} 0, & \text{for} & y<s, \\ 1, & \text{for} & y>s. \end{matrix}\right\} s>0. \quad . \quad . \quad (7)$$

With the function $F_s(y)$, defined on the right-hand side of (7), we have from (6)

$$H(x, y)=\frac{1}{(n-1)!}\int_0^x F_s(y)\,s^{n-1}\,ds. \quad . \quad . \quad (8)$$

If we wish to know $H(x, y)$ for a set of variables x, y, where $x<y$, we have in $F_s(y)$ of (8) always $s<x<y$, and therefore from (7)

$$F_s(y)=1 \quad \text{for} \quad s<x<y,$$

so that

$$H(x, y)=\frac{1}{(n-1)!}\int_0^x s^{n-1}\,ds=\frac{x^n}{n!} \quad \text{for} \quad x<y.$$

Seeking the value of $H(x, y)$ in the case $y<x$, we have

$$F_s(y)=1 \quad \text{for} \quad s<y,$$

$$F_s(y)=0 \quad \text{for} \quad y<s<x,$$

so that (6) becomes here

$$H(x, y)=\frac{1}{(n-1)!}\int_0^y s^{n-1}\,ds=\frac{y^n}{n!} \quad \text{for} \quad y<x.$$

We can state the result in the following way:—

$$\frac{1}{(p+q)^n} \doteqdot H(x, y)=\left\{\begin{matrix} \dfrac{x^n}{n!} & \text{for} & x<y, \\ \\ \dfrac{y^n}{n!} & \text{for} & y<x. \end{matrix}\right\}. \quad . \quad . \quad (9)$$

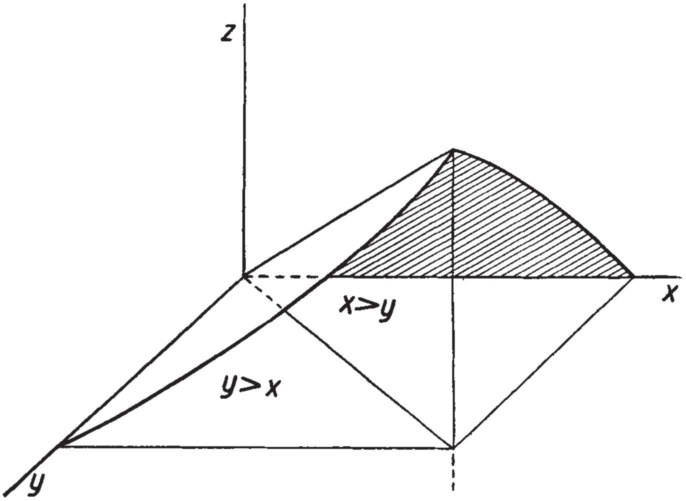
For $n=1$, $H(x,y)$ would become a kind of a "roof-function" when we measure $H(x, y)$ along the ordinate z above the region for positive values of x and y (see figure).

This geometrical interpretation is only given as an illustration, but will not be used more in what follows, where we wish again to draw attention entirely to the analytical side of the problem.

2. We will now consider an application of the simultaneous operational calculus to the evaluation of some integrals.

Suppose we require the value of

$$S=\int_0^\infty J_n^2(u)\frac{du}{u}. \quad \dots\dots \quad (10)$$



Writing

$$u=2\sqrt{\xi}, \quad \dots\dots\dots \quad (11)$$

we wish to know

$$2S=\int_0^\infty J_n(2\sqrt{\xi})J_n(2\sqrt{\xi})\frac{1}{\xi}d\xi.$$

The reason for this substitution (11) is to bring the Bessel function and its argument in a form suitable for operational interpretation. In I. form. (29), namely, it is proved that when p and x belong together in

$$x \doteq \frac{1}{p},$$

the following operational transformation exists :

$$x^{\frac{n}{2}} J_n(2\sqrt{x}) \doteqdot p^{-n} e^{-\frac{1}{p}}. \quad . \quad . \quad . \quad . \quad (12)$$

From the above it will be clear that we better write

$$2S = \int_0^\infty \frac{\xi^{\frac{n}{2}} J_n(2\sqrt{\xi}) \, . \, \xi^{\frac{n}{2}} J_n(2\sqrt{\xi})}{\xi^{n+1}} \, d\xi.$$

Now for what follows the essential point is that we do not transform a function of ξ into a function of p, but that we first introduce new variables α and β, not instead of ξ, but entirely apart from ξ, namely, by constructing the function

$$S_{\alpha, \beta} = \int_0^\infty \frac{(\alpha\xi)^{\frac{n}{2}} J_n(2\sqrt{\alpha\xi}) \, (\beta\xi)^{\frac{n}{2}} J_n(2\sqrt{\beta\xi})}{\xi^{n+1}} \, d\xi. \quad (13)$$

For $\alpha = \beta = 1$ we have

$$S_{1,1} = 2S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

To evaluate $S_{\alpha, \beta}$ as a function of α and β we will use our operational methods :

a function of α will be transformed into a function of p,

$$\alpha \doteqdot \frac{1}{p},$$

and a function of β into another of q,

$$\beta \doteqdot \frac{1}{q}.$$

From the mentioned theorem (see (12), where p and x belong together) we know

$$\alpha^{\frac{n}{2}} J_n(2\sqrt{\alpha}) \doteqdot p^{-n} e^{-\frac{1}{p}}, \quad . \quad . \quad . \quad . \quad (15)$$

$$\beta^{\frac{n}{2}} J_n(2\sqrt{\beta}) \doteqdot q^{-n} e^{-\frac{1}{q}}. \quad . \quad . \quad . \quad . \quad (16)$$

In these operational relations ξ can be treated as a constant.

Now, using for an instance the notation of I. form. (5), it follows from

$$f(p) \doteqdot h(x)$$

that

$$f\left(\frac{p}{s}\right) \doteqdot h(sx),$$

if s is a positive number.

Applying this with $s=\xi$ and $x=\alpha$ or β, we find from (15, 16)

$$(\alpha\xi)^{\frac{n}{2}}J_n(2\sqrt{\alpha\xi}) \doteqdot \left(\frac{p}{\xi}\right)^{-n} e^{-\frac{\xi}{p}}, \quad . \quad . \quad (15\,a)$$

$$(\beta\xi)^{\frac{n}{2}}J_n(2\sqrt{\beta\xi}) \doteqdot \left(\frac{q}{\xi}\right)^{-n} e^{-\frac{\xi}{q}}. \quad . \quad . \quad . \quad (16\,a)$$

Therefore from (13)

$$S_{\alpha,\beta} \doteqdot \int_0^\infty \frac{\left(\frac{p}{\xi}\right)^{-n} e^{-\frac{\xi}{p}} \left(\frac{q}{\xi}\right)^{-n} e^{-\frac{\xi}{q}}}{\xi^{n+1}}\, d\xi$$

$$= \frac{1}{p^n q^n}\int_0^\infty e^{-\left(\frac{1}{p}+\frac{1}{q}\right)\xi}\, \xi^{n-1}\, d\xi.$$

The symbol $\doteqdot$ contains here the $\frac{1}{p} \doteqdot \alpha$ as well as the $\frac{1}{q} \doteqdot \beta$ tranformation.

Substituting

$$\left(\frac{1}{p}+\frac{1}{q}\right)\xi = s,$$

we have

$$S_{\alpha,\beta} \doteqdot \frac{1}{(p+q)^n}\int_0^\infty e^{-s}s^{n-1}ds = \frac{\Pi(n-1)}{(p+q)^n}.$$

For the evaluation of $S_{\alpha,\beta}$ we need, therefore, the interpretation of $\frac{1}{(p+q)^n}$, which was found in the foregoing paragraph.

Applying (9), we obtain for $S_{\alpha,\beta}$

$$S_{\alpha,\beta} = \left\{ \begin{array}{ll} \frac{\Pi(n-1)}{\Pi(n)}\alpha^n, & \text{if} \quad \alpha<\beta, \\ \frac{\Pi(n-1)}{\Pi(n)}\beta^n, & \text{if} \quad \beta<\alpha, \end{array} \right\} \quad . \quad . \quad . \quad (17)$$

and

$$S_{1,1} = \frac{1}{n}.$$

On account of (10) and (14) we find, therefore, for the required value of the integral S:

$$S = \int_0^\infty J_n^2(u)\,\frac{du}{u} = \frac{1}{2n}.$$

Meanwhile we found even more than that, since from (13) and (17) it follows that

$$\int_0^\infty \frac{J_n(2\sqrt{\alpha\xi})J_n(2\sqrt{\beta\xi})}{\xi}\,d\xi = \begin{cases} \dfrac{1}{n}\left(\dfrac{\alpha}{\beta}\right)^{\frac{n}{2}}, & \text{if } \alpha<\beta, \\[2ex] \dfrac{1}{n}\left(\dfrac{\beta}{\alpha}\right)^{\frac{n}{2}}, & \text{if } \beta<\alpha. \end{cases}$$

If we substitute
$$\xi=\frac{t^2}{4},$$
$$\alpha=a^2, \qquad \beta=b^2,$$
we obtain a special case of the discontinuous integral of Weber and Schafheitlin (comp. Watson, 'Bessel Functions,' § 13, 42), namely

$$\int_0^\infty \frac{J_n(at)J_n(bt)}{t}\,dt = \begin{cases} \dfrac{1}{2n}\left(\dfrac{a}{b}\right)^{n}, & \text{if } a<b, \\[2ex] \dfrac{1}{2n}\left(\dfrac{b}{a}\right)^{n}, & \text{if } b<a. \end{cases}$$

3. Another application of simultaneous operational calculus can be made in the proof of the orthogonality relation for Bessel functions of the same order.

As is known, this relation is

$$\int_0^1 uJ_n(au)J_n(bu)\,du=0,$$

where a and b are two different roots of

$$J_n(x)=0.$$

This has been found from the following equation :—

$$(a^2-b^2)\int_0^1 uJ_n(au)J_n(bu)\,du = aJ_n(b)J_{n+1}(a)-bJ_n(a)J_{n+1}(b). \quad . \quad . \quad (18)$$

This relation can be proved by operational methods as follows. Substituting

$$a=\sqrt{\alpha}, \quad b=\sqrt{\beta}, \quad u=2\sqrt{\xi},$$

we have to prove

$$(\alpha-\beta)\int_0^{\frac{1}{4}} \frac{(\alpha\xi)^{\frac{n}{2}}J_n(2\sqrt{\alpha\xi})\,(\beta\xi)^{\frac{n}{2}}J_n(2\sqrt{\beta\xi})}{\xi^n}\,d\xi$$
$$=\tfrac{1}{2}\alpha^{\frac{n+1}{2}}J_{n+1}(\sqrt{\alpha})\beta^{\frac{n}{2}}J_n(\sqrt{\beta})-\tfrac{1}{2}\alpha^{\frac{n}{2}}J_n(\sqrt{\alpha})\beta^{\frac{n+1}{2}}J_{n+1}(\sqrt{\beta}). \quad . \quad . \quad . \quad (19)$$

If we relate

$$\alpha \doteqdot \frac{1}{p},$$

$$\beta \doteqdot \frac{1}{q},$$

and use again the equations (15 a) and (16 $'a$), the right-hand side of (19) is transformed into

$$\frac{e^{-\frac{1}{4p}-\frac{1}{4q}}}{2^{2n+2}p^n q^n}\left(\frac{1}{p}-\frac{1}{q}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

For the transformation of the left-hand side of (19) we need the rule mentioned in I. form. (12), namely, it follows from

$$f(p) \doteqdot h(x),$$

that

$$\left(-p\frac{d}{dp}\right)^n f(p) \doteqdot x^n h(x), \qquad n>0.$$

This gives in our case ($n=1$, $x=\alpha$ or β) :

$$\alpha(\alpha\xi)^{\frac{n}{2}} J_n(2\sqrt{\alpha\xi}) \doteqdot -p\frac{d}{dp}\left(\frac{\xi^n e^{-\frac{\xi}{p}}}{p^{n+1}}\right),$$

$$\beta(\beta\xi)^{\frac{n}{2}} J_n(2\sqrt{\beta\xi}) \doteqdot -q\frac{d}{dq}\left(\frac{\xi^n e^{-\frac{\xi}{q}}}{q^{n+1}}\right),$$

so that the left-hand side of (19) becomes

$$\frac{n+1}{p^n q^n}\left(\frac{1}{p}-\frac{1}{q}\right)\int_0^{\frac{1}{4}} \xi^n e^{-\frac{\xi}{p}-\frac{\xi}{q}}\,d\xi - \frac{1}{p^n q^n}\left(\frac{1}{p^2}-\frac{1}{q^2}\right)\int_0^{\frac{1}{4}} \xi^{n+1} e^{-\frac{\xi}{p}-\frac{\xi}{q}}\,d\xi.$$
$$. \quad . \quad . \qquad (21)$$

The equality between (20) and (21) is directly proved by introducing

$$s=\frac{1}{p}+\frac{1}{q},$$

and hence the required proof of (18) is given in an operational way.

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